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Crossover exponents in a superconductor–nonlinear-normal-conductor network below the percolation threshold

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Abstract. The crossover behaviour in a d-dimensional percolating superconductor (S)–nonlinear-normal-conductor (N) random network is studied. The system is composed of a volume fraction f of superconductors, and fraction $1 - f$ of nonlinear normal conductors with the current (i)–voltage (v) relation of the form $i = g_1 v + \chi_1 v^{\beta+1}$, where g_1 and χ_1 are the linear and nonlinear response of normal conductors. As the percolating threshold f_c of the superconductor is approached from below, the crossover electric field $|\vec{E}_{L-NL}|$ and corresponding current density $|\vec{J}_{L-NL}|$, defined as the field and current density at which the linear and nonlinear response of the random network become comparable, are found to have power-law dependence $|\vec{E}_{L-NL}| \sim (f_c - f)^{M(\beta)}$, $|\vec{J}_{L-NL}| \sim (f_c - f)^{N(\beta)}$ respectively. Within the effective medium approximation (EMA), critical exponents $M(\beta)$ and $N(\beta)$ are estimated to be $\frac{1}{2}$ and $-\frac{1}{2}$ for all spatial dimensions d and arbitrary nonlinearity β. By means of the multifractal approach, explicit expressions for $M(\beta)$ and $N(\beta)$ as a function of β are obtained. For $d = 2$, we investigate the influence of nonlinearity β on the crossover properties analytically and numerically; while for $d = 3$, we present such special values as $M(2) \approx 0.74$ and $N(2) \approx -0.01$; $M(4) \approx 0.79$ and $N(4) \approx 0.04$; $N(\infty) \approx 0.88$ and $N(\infty) \approx 0.13$. Careful examination of exponents $M(\beta)$ and $N(\beta)$ gives interesting crossover behaviour. Numerical results are also compared with previous bounds and good agreement is found.

1. Introduction

Nonlinear inhomogeneous composite materials have attracted much interest in recent years [1–3]. Typically, such a system consists of a material with weakly nonlinear current (i)–voltage (v) characteristics of the form $i = gv + \chi v^3$, embedded in a linear or nonlinear host. Two basic questions concerning such random systems are often raised: one is the calculation of the effective nonlinear response [4–7]; the other is critical properties and scaling behaviour of linear and nonlinear properties near the percolation threshold [4, 8–10].

For the second question, the percolation theory [11] has been extremely useful in describing linear properties in a superconductor–normal-conductor random network (S/N limit) and a normal-conductor–insulator system (N/I limit) near the percolation threshold. According to the theory of percolation, one can define the critical exponent s (or t) to describe the divergence of the linear conductivity (or resistivity). Such divergence is geometrical in nature and is related to the fractal character of the incipient infinite cluster. As to nonlinear properties, Stroud and Hui [4] demonstrated the relation between the nonlinear–random-network problem and the noise problem in the corresponding linear system, and obtained the

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critical exponent of the effective third-order susceptibility in a nonlinear normal-conductor–insulator (N/I) composite; Blumenfeld and Bergman [12] pointed out that the results in [4] can be used to derive a characteristic value of the current (crossover current) at which the linear and nonlinear response in N/I mixtures become comparable. For superconductor–nonlinear-normal-conductor (S/N) mixtures below the percolation threshold of the superconductor, Hui defined the crossover current density $|\vec{J}_{L-NL}|$ to characterize the crossover effect from linearity to nonlinearity within the 'nodes–links–blobs' (NLB) picture [13]; later, he also studied the analogous problem and derived a value of the crossover electric field $|\vec{E}_{L-NL}|$ which marks the transition from linear to nonlinear behaviour [14]. Critical exponents of these nonlinear physical parameters are evaluated by means of various methods including the effective medium approximation (EMA) [8, 10, 14], numerical simulation [9, 10] and the relation to the noise exponents [10, 14]. All these considerations lead to the same conclusion, that, due to geometric effects near the percolation threshold, the linear regime in the i–v response shrinks. In other words, the nonlinear response becomes increasingly important as the percolation threshold is approached. Such a percolating S/N system will be of potential practical use, because it has a high conductivity and yet is highly nonlinear [13].

To our knowledge, much work concentrates on the critical behaviour of effective cubic nonlinearity, i.e., $\beta = 2$. In fact, β is not limited to 2 generally. For the material without inversion symmetry, the lowest-order nonlinearity is $\beta = 1$. For example, the i–v characteristic of a carbon–wax mixture is found to be nonlinear and the leading nonlinear term is quadratic (or $\beta = 1$) [15]; numerical simulation of the effective high-order nonlinearity with $\beta = 2, 3, 4, 6$ of a random network has also been investigated [16, 17]; Fu and Resca [18] have considered a non-integer $\beta = 2.5$ to discuss the effective response; higher nonlinearity may be found in some ceramic two-dimensional materials at low temperature [19], as in a recent experiment on laser-irradiated polymers [20] and condensed matter to which a sufficiently strong field is applied [12]; theoretical studies on the effective nonlinear response with arbitrary nonlinearity have been proposed [16, 17] also. The purpose of this paper is twofold: we firstly generalize the previous studies on the critical behaviour of effective nonlinear properties [9, 10, 14], which is only applicable to cubic nonlinearity, to the case of arbitrary nonlinearity, i.e., the normal conductor has the weak nonlinear form: $i = gv + \chi_1 v^{\beta+1}$, $\beta > 0$. Secondly, both the crossover field $|\vec{E}_{L-NL}|$ and corresponding current density $|\vec{J}_{L-NL}|$ are examined at the same time. Previous work has often concentrated on $|\vec{E}_{L-NL}|$ in a percolating S/N system with third-order nonlinearity. The effect of nonlinearity β on these crossover physical parameters in the percolating S/N random network is examined systematically. Below the percolation threshold f_c of the superconductor, the magnitudes of crossover field $|\vec{E}_{L-NL}|$ and current density $|\vec{J}_{L-NL}|$ are found to have power-law dependence $|\vec{E}_{L-NL}| \sim (f_c - f)^{M(\beta)}$, $|\vec{J}_{L-NL}| \sim (f_c - f)^{N(\beta)}$ respectively. $M(\beta)$ and $N(\beta)$ are critical exponents of these crossover physical parameters and are dependent on β in general. A crude estimate can be obtained by using EMA [16], which gives $M(\beta) = \frac{1}{2}$ and $N(\beta) = -\frac{1}{2}$ for all spatial dimensions d and arbitrary nonlinear exponent β. With the multifractal approach [21], the dependence of explicit expressions of $M(\beta)$ and $N(\beta)$ on the nonlinearity β is obtained. A detailed investigation of $M(\beta)$ and $N(\beta)$ as a function of β is made for a two-dimensional percolating S/N system, and numerical results for $d = 3$ are presented for some special nonlinear orders. Theoretical predictions are compared with previous work also [10, 13, 14].

2. Estimate of exponents $M(\beta)$ and $N(\beta)$

We consider a d-dimensional superconductor–nonlinear-normal-conductor hypercubic random network with the volume fraction f of superconductors and with $1 - f$ of weakly nonlinear normal conductors. Normal conductors have a current–voltage response of the form $i = g_1 v + \chi_1 v^{\beta+1}$, where g_1 and χ_1 are the linear conductance and the nonlinear response, while superconductors have $g_2 = \infty$. Previous studies assumed third-order nonlinearity, namely $\beta = 2$. Such cubic nonlinearity is the lowest-order nonlinearity appearing in a material with inversion symmetry [4–8]. Throughout this work, the nonlinear term is assumed to be weak, i.e., $\chi_1 v^{\beta}/g_1 \ll 1$. We concentrate on the regime where $f < f_c$. For $f > f_c$, the superconductor forms a connected path across the system and the whole system becomes perfectly conducting. Below the percolation threshold of the superconductor $f < f_c$, the effective nonlinear response of the whole system can be represented by

$$I = G_e V_0 + A_e V_0^{\beta+1} \tag{1}$$

where G_e and A_e are the effective bulk linear conductance and nonlinear response respectively [22], and I is the current across the whole network when the external voltage V_0 is applied to the system. Equivalently, each bond is replaced by a conductor with the form

$$i = g_e v + \chi_e v^{\beta+1}. \tag{2}$$

Note that $g_e \propto G_e$ and $\chi_e \propto A_e$, while G_e and A_e are given by

$$G_e = g_1 \sum_c \left(\frac{V_c}{V_0}\right)^2 \tag{3}$$

and

$$A_e = \chi_1 \sum_c \left(\frac{V_c}{V_0}\right)^{(\beta+2)}. \tag{4}$$

Here the summation is performed over all the normal conductors as no voltage drop is across the superconductor phase; V_c is the voltage drop in the normal conductor in the corresponding linear random system (obtained by solving the same inhomogeneous problem with $\chi_1 = 0$), when the external voltage V_0 is imposed on the network. Equation (4) gives an expression for A_e to the first order in the nonlinear response χ_1.

The crossover voltage V_{L-NL} is defined as the voltage at which the linear and the nonlinear response become comparable and can be obtained by equating the two terms on the right-hand side of equation (1); we have

$$V_{L-NL} = \left(\frac{G_e}{A_e}\right)^{1/\beta}. \tag{5}$$

The corresponding crossover current then can be obtained by using the value of V_{L-NL} in equation (1) and is given by

$$I_{L-NL} = 2G_e V_{L-NL} = 2G_e \left(\frac{G_e}{A_e}\right)^{1/\beta}. \tag{6}$$

We note that, in order to determine critical exponents, it is wise to use not V_{L-NL} and I_{L-NL}, but rather the more universal quantities which are independent of the size of the network L, in the thermodynamic limit $L \to \infty$. Therefore, we are interested in crossover physical parameters such as the crossover electric field $|\vec{E}_{L-NL}|$ and the crossover current density $|\vec{J}_{L-NL}|$, which are defined as

$$|\vec{E}_{L-NL}| = V_{L-NL}/L = \left(\frac{G_e}{A_e}\right)^{1/\beta} /L \propto \left(\frac{g_e}{\chi_e}\right)^{1/\beta} \tag{7}$$

$$|\vec{J}_{L-NL}| = I_{L-NL}/L^{d-1} \propto g_e \left(\frac{g_e}{\chi_e}\right)^{1/\beta}. \tag{8}$$

Below the percolation threshold of the superconductor f_c, crossover physical parameters $|\vec{E}_{L-NL}|$ and $|\vec{J}_{L-NL}|$ are found to behave as

$$|\vec{E}_{L-NL}| \sim (f_c - f)^{M(\beta)} \tag{9}$$

and

$$|\vec{J}_{L-NL}| \sim (f_c - f)^{N(\beta)} \tag{10}$$

respectively, where $M(\beta)$ and $N(\beta)$ are critical exponents of crossover physical parameters. In the following subsection, we shall show how to obtain these critical exponents and analyse the critical behaviour of $|\vec{E}_{L-NL}|$ and $|\vec{J}_{L-NL}|$ near f_c.

2.1. An effective medium approximation (EMA)

An EMA [5, 8] has been proposed for estimating the effective nonlinear response in a third-order nonlinear random mixture. Later, it is also generalized to calculate the effective nonlinear response for the $(\beta + 1)$th-order nonlinear mixture [16]. Here we give a brief review.

For the system we studied, the effective arbitrary order nonlinear response χ_e can be written as [16]

$$\chi_e = \frac{(1-f)\chi_1 \langle V_c^{\beta+2}\rangle}{V_0^{\beta+2}} \tag{11}$$

where $\langle \cdots \rangle$ denotes the spatial average. By means of a decoupling scheme [6, 16], $\langle V_c^{\beta+2}\rangle$ can be written approximately as

$$\langle V_c^{\beta+2}\rangle \approx \langle V_c^2\rangle^{\frac{\beta+2}{2}}. \tag{12}$$

As mentioned in [16], the approximation is based on the assumption that the fluctuations in the voltage $\langle V_c^{\beta+2}\rangle - \langle V_c^2\rangle^{\frac{\beta+2}{2}}$ within the normal conductor are small compared to $\langle V_c^{\beta+2}\rangle$ itself. This approximation will be accurate in geometries for which the voltage drop is nearly uniform within the nonlinear normal conductors and less accurate when these fluctuations are large as in a random mixture near the percolation threshold.

It is known that $\langle V_c^2\rangle$ can be expressed as [4]

$$\langle V_c^2\rangle = \frac{1}{1-f}\frac{\partial g_e}{\partial g_1} V_0^2. \tag{13}$$

Substituting equations (12) and (13) into equation (11), we can arrive at a compact formula for the effective nonlinear response χ_e

$$\chi_e = \frac{\chi_1}{(1-f)^{\beta/2}}\left(\frac{\partial g_e}{\partial g_1}\right)^{\frac{\beta+2}{2}}. \tag{14}$$

The EMA is completed by calculating g_e from some linear approximations such as the Maxwell–Garnett formula and linear EMA. The linear EMA is a self-consistent scheme which is valid for the whole fraction regime and reads

$$\sum_i f_i \frac{g_i - g_e}{g_i + (d-1)g_e} = 0 \tag{15}$$

where d is the dimension of the system while f_i is the fraction of the ith component. For the S/N limit, the effective linear and $(\beta + 1)$th nonlinear response can be obtained as follows:

$$g_e = \frac{g_1}{d}\left(\frac{1}{d} - f\right)^{-1} \tag{16}$$

$$\chi_e = \frac{\chi_1}{d^{\frac{\beta+2}{2}}}\frac{1}{(1-f)^{\frac{\beta}{2}}}\left(\frac{1}{d} - f\right)^{-\frac{\beta+2}{2}}. \tag{17}$$

Then, the crossover field $|\vec{E}_{L-NL}|$ and the corresponding current density $|\vec{J}_{L-NL}|$ are

$$|\vec{E}_{L-NL}| \propto \left(\frac{g_e}{\chi_e}\right)^{\frac{1}{\beta}} \sim (f_c - f)^{\frac{1}{2}} \tag{18}$$

$$|\vec{J}_{L-NL}| \propto g_e \left(\frac{g_e}{\chi_e}\right)^{\frac{1}{\beta}} \sim (f_c - f)^{-\frac{1}{2}}. \tag{19}$$

Thus within the EMA, $f_c = 1/d$, and crossover exponents $M(\beta) = \frac{1}{2}$ and $N(\beta) = -\frac{1}{2}$ for all spatial dimensions d and arbitrary nonlinearity β, i.e., these crossover parameters behave the same way independent of the detail of the nonlinear behaviour of individual components. Results also show that in a percolating S/N system, the crossover field vanishes while the crossover current diverges. This requires that we should not introduce the crossover current density but the crossover electric field to characterize the crossover behaviour, because the former $|\vec{J}_{L-NL}| \to +\infty$ as $f \to f_e^-$, which cannot be easily accessed in experiments.

EMA may give the correct behaviour near the percolation threshold qualitatively but usually predicts the incorrect exponents, because it does not take into account the full complexity of the spatial fluctuations of the voltage to which the effective nonlinear properties are so sensitive.

2.2. *A multifractal approach*

In order to describe multifractal properties of a S/N random network, one often defines the multifractal moment of the voltage distributions for normal-conductor bonds [21]

$$T_p = \sum_c \left(\frac{V_c}{V_0}\right)^{2p}. \tag{20}$$

It turns out that various moments of T_p have different physical interpretations for different p. For instance, the zero moment ($p = 0$) describes the fractal dimensionality of the backbone, the second moment ($p = 1$) is proportional to the bulk conductance G_e and the fourth ($p = 2$) is closely related to the $1/f$ noise exponent while the infinite moment is governed by the so-called single disconnected bonds.

For $f < f_c$ and in the thermodynamic limit ($L \to \infty$), T_p depends not only on L but also on $(f_c - f)$ and scales as a power law [21]

$$T_p \sim L^{d-2p}(f_c - f)^{[s(2p)-2ps(2)]}. \tag{21}$$

Here $s(2)$ is nothing but the critical exponent of the effective linear conductivity in the S/N system. According to the definition of G_e and A_e, we have

$$G_e \sim L^{d-2}(f_c - f)^{-s(2)} \tag{22}$$

$$A_e \sim L^{d-(\beta+2)}(f_c - f)^{[s(\beta+2)-(\beta+2)s(2)]}. \tag{23}$$

Thus, crossover physical parameters behave as

$$|\vec{E}_{L-NL}| = \left(\frac{G_e}{A_e}\right)^{1/\beta} \Big/ L \sim (f_c - f)^{\frac{(\beta+1)s(2)-s(\beta+2)}{\beta}} \tag{24}$$

and

$$|\vec{J}_{L-NL}| = 2G_e|\vec{E}_{L-NL}|/L^{d-2} \sim (f_c - f)^{\frac{s(2)-s(\beta+2)}{\beta}}. \tag{25}$$

Here we emphasize that the above relations are correct for $L > \xi$ (ξ is the correlation length and $\xi \sim (f_c - f)^{-\nu}$ near the percolation threshold), i.e., in the Euclidean regime, and thus hold in the thermodynamic limit ($L \to \infty$) and $f < f_c$. Therefore, critical exponents $M(\beta)$

and $N(\beta)$, which describe the dependence of the crossover electric field and current density on $f_c - f$ in the S/N limit, are given explicitly as follows:

$$M(\beta) = \frac{(\beta+1)s(2) - s(\beta+2)}{\beta} \tag{26}$$

and

$$N(\beta) = \frac{s(2) - s(\beta+2)}{\beta}. \tag{27}$$

In order to investigate critical properties of $M(\beta)$ and $N(\beta)$, we must look for the expressions for $s(\beta+2)$.

For a finite system of size L right at $f = f_c$, the correlation length diverges and the whole system is in the fractal and self-similar region. In this case, T_p will only depend on L because the correlation length ξ is limited by the finite size L and $\xi \approx L$. Then the dependence of T_p and L can be obtained by putting $(f_c - f) \sim \xi^{-1/\nu} = L^{-1/\nu}$, that is

$$T_p \sim L^{\frac{(d-2p)\nu - s(2p) + 2ps(2)}{\nu}}. \tag{28}$$

On the other hand, at $f = f_c$, the voltage moment T_p can also scale as [23]

$$T_p \sim L^{\frac{\zeta(2p)}{\nu}} \tag{29}$$

where $\zeta(2p)$ is the moment exponent. Comparing with the above two equations, we can obtain the relation between $s(2p)$ and $\zeta(2p)$

$$s(2p) - 2ps(2) = (d-2p)\nu - \zeta(2p). \tag{30}$$

Letting $2p = \beta + 2$ and substituting equation (30) into equations (26) and (27), we can easily obtain

$$M(\beta) = \nu + \frac{\zeta(\beta+2) - \zeta(2)}{\beta} \tag{31}$$

and

$$N(\beta) = (d-1)\nu + \frac{\zeta(\beta+2) - (\beta+1)\zeta(2)}{\beta}. \tag{32}$$

So far, these expressions represent an exact scaling law. Here, we have formulated these crossover exponents as a function of β based on the multifractal approach. Critical exponents of the crossover electric field and the corresponding current density can readily be analysed and calculated by use of equations (31) and (32).

2.3. Numerical discussions

Crossover exponents $M(\beta)$ and $N(\beta)$ in a percolating S/N system become very important when one wants to know how nonlinear properties are dependent on β.

First, we analyse the properties of exponents $M(\beta)$ and $N(\beta)$ in the two-dimensional case. In this case, from the duality consideration, it has been shown [23] that $\zeta(\beta+2)$ is consistent with the moment exponent in a two-dimensional percolating N/I network and has the form

$$\zeta(\beta+2) = (\beta+2)p(2) - p(\beta+2) \tag{33}$$

with $p(\beta+2) \equiv \beta + 1 + \{(\beta+2)\ln(5/4) - \ln[1 + 2^{-(\beta+2)}]\}/\ln 2$ [23].

According to equations (31)–(33), we obtain the first limit case that $\beta \to 0^+$

$$\lim_{\beta\to 0^+} M(\beta) = \nu - \lim_{\beta\to 0^+} \frac{2^{-(\beta+2)}}{1 + 2^{-(\beta+2)}} \approx 1.13 \tag{34}$$

and

$$\lim_{\beta\to 0^+} N(\beta) = (d-2)\nu - \zeta(2) + \lim_{\beta\to 0^+} M(\beta) = -0.19. \tag{35}$$

Crossover exponents for $\beta = 2$, on which previous work mainly concentrates [10, 14], become

$$M(\beta = 2) = 1.22 \qquad \text{and} \qquad N(\beta = 2) = -0.10. \tag{36}$$

In the other limit $\beta \to +\infty$, we give

$$\lim_{\beta\to +\infty} M(\beta) = \nu = 4/3 \tag{37}$$

and

$$\lim_{\beta\to +\infty} N(\beta) = (d-1)\nu - \zeta(2) = 0.01. \tag{38}$$

In deriving the above results, we have used $\nu = \frac{4}{3}$ for $d = 2$.

Perhaps the most important properties are the monotonicity

$$\frac{\mathrm{d}M(\beta)}{\mathrm{d}\beta} = \frac{\ln(5/4) - \ln[1 + 2^{-(\beta+2)}] - \beta \ln 2/(1 + 2^{(\beta+2)})}{\beta^2 \ln 2}. \tag{39}$$

Note that the denominator is positive and the numerator $f(\beta) \equiv \ln(5/4) - \ln[1 + 2^{-(\beta+2)}] - \beta \ln 2/(1 + 2^{(\beta+2)})$ in equation (39) is always larger than zero for $\beta > 0$ because (i) $f(\beta \to 0^+) \to 0^+$; (ii) $f(\beta \to +\infty) = \ln(5/4)$ and (iii) $\mathrm{d}f(\beta)/\mathrm{d}\beta = 2^{(\beta+2)}\beta(\ln 2)^2/[1+2^{(\beta+2)}]^2 > 0$. We then have

$$\frac{\mathrm{d}M(\beta)}{\mathrm{d}\beta} > 0 \tag{40}$$

and

$$\frac{\mathrm{d}N(\beta)}{\mathrm{d}\beta} = \frac{\mathrm{d}M(\beta)}{\mathrm{d}\beta} > 0 \tag{41}$$

for arbitrary $\beta > 0$.

Equations (34), (37) and (40) mean that $M(\beta) > 0$ for all $\beta > 0$, thus for $d = 2$, $|\vec{E}_{L-NL}| \sim (f_c - f)^{M(\beta)} \to 0$ as $f \to f_c^-$, that is to say, the crossover electric field will vanish for definite $f_c - f$. This corresponds to an enhancement in the nonlinear response near the percolation threshold relative to a system consisting only of the nonlinear components, and it implies that the region of linear response shrinks and a small electric field is enough to lead to an appreciable nonlinear response near the percolation threshold.

Because $\mathrm{d}M(\beta)/\mathrm{d}\beta > 0$, $M(\beta)$ increases with increasing β. This means that the crossover field $|\vec{E}_{L-NL}|$ vanishes faster for larger β, thus for larger nonlinearity β, a somewhat smaller electric field is needed to stimulate an appreciable nonlinear response, and the larger the region of nonlinear response may be correspondingly. In contrast, in the system with small β such as $\beta \to 0^+$, $|\vec{E}_{L-NL}| \sim (f_c - f)^{1.13}$ takes the maximum for definite $f_c - f$, thus it possesses the largest linear region. This can be well understood because as $\beta \to 0^+$, the nonlinear term $\chi_1 v^{\beta+1}$ of the nonlinear component becomes $\chi_1 v$ and the nonlinear normal conductor has a linear i–v response $i = (g_1 + \chi_1)v$.

As to the critical exponent $N(\beta)$, according to equation (41), $N(\beta)$ increases with increasing β also. Combining equation (35), which gives $N(\beta) < 0$, with equation (38), which gives $N(\beta) > 0$, we can predict that $N(\beta)$ may take negative, zero and positive values with increasing β, thus $|\vec{J}_{L-NL}|$ has

$$\begin{array}{lll} |\vec{J}_{L-NL}| \to +\infty & N(\beta) < 0 & \beta < \beta_c \\ |\vec{J}_{L-NL}| \to \text{ constant} & N(\beta) = 0 & \beta = \beta_c \\ |\vec{J}_{L-NL}| \to 0 & N(\beta) > 0 & \beta > \beta_c \end{array} \tag{42}$$

where $\beta_c \approx 28$ is a critical value at which the exponent $N(\beta) = 0$.

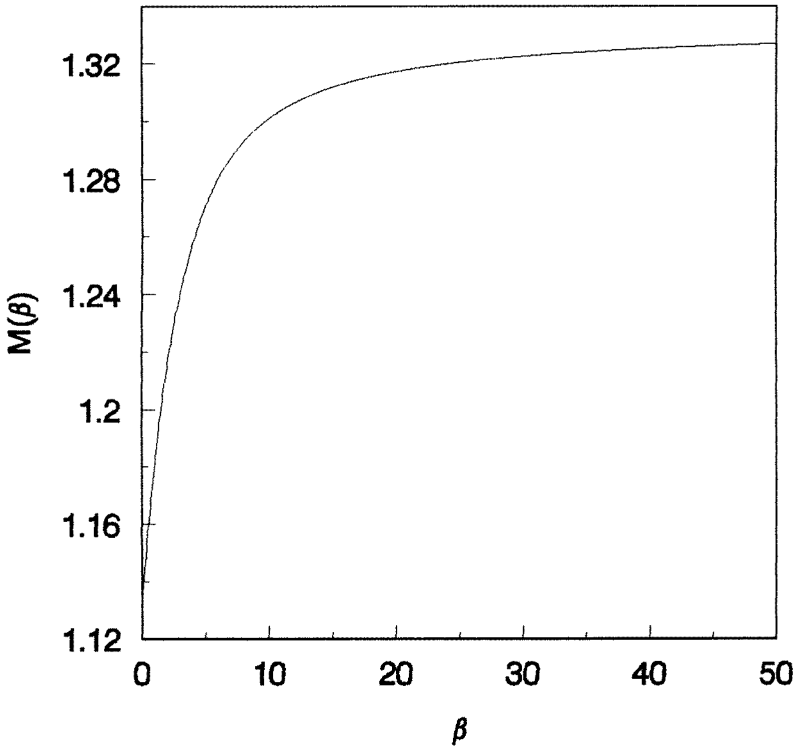

Figure 1. For a two-dimensional percolating S/N system, the critical exponent $M(\beta)$ of the crossover electric field $|\vec{E}_{L-NL}|$ as a function of β based on equation (31).

Numerical results for exponents $M(\beta)$ and $N(\beta)$ in the two-dimensional case based on equations (31) and (32) are shown in figures 1 and 2 respectively. We can clearly find our analytical predictions, i.e., $M(\beta) > 0$ and $N(\beta)$ takes negative, zero and positive values with increasing β; $M(\beta)$ and $N(\beta)$ are monotonically increasing functions of β.

Thus for $d = 2$, the qualitative analysis and numerical results of $M(\beta)$ and $N(\beta)$ show the following. (i) For small $\beta(\beta < \beta_c)$, only $|\vec{E}_{L-NL}|$ can describe the crossover effect. It is trivial to use $|\vec{J}_{L-NL}|$ to characterize the crossover behaviour, because $|\vec{J}_{L-NL}|$ diverges as $f \to f_c^-$ and cannot be experimentally accessed. Such conclusions demonstrate that it is more appropriate to consider $|\vec{E}_{L-NL}|$ than $|\vec{J}_{L-NL}|$ in the case of $\beta = 2 < \beta_c$ [14] (also see equation (36)). (ii) For large $\beta(\beta > \beta_c)$, both $|\vec{E}_{L-NL}|$ and $|\vec{J}_{L-NL}|$ vanish as $f \to f_c^-$, and thus can be applied to describe crossover effects also. This is an interesting and a new result, which has not been reported because previous work only studies the cubic nonlinear response.

Then, we investigate the case of $d = 3$. Because of lack of the explicit expression for $\zeta(\beta+2)$ in this case, we can only present some concrete numerical results such as $M(\beta = 2) \approx 0.74$ and $N(\beta = 2) = -0.01$; $M(\beta = 4) \approx 0.79$ and $N(4) \approx 0.04$; $M(\beta \to +\infty) = 0.88$, $N(\beta \to +\infty) \approx 0.13$ with the aim of numerical results of $\zeta(\beta + 2)$ [21]. According to the above results, we predict that the critical behaviour of $|\vec{J}_{L-NL}|$ and $|\vec{E}_{L-NL}|$ in $d = 3$ may take on similar behaviour as that in $d = 2$. However, the crossover field in $d = 2$ may vanish faster than that in $d = 3$ as $f \to f_c^-$ because of its larger critical exponents, for example, $M(\beta = 2) = 1.22(2D) > 0.74(3D)$; $M(\beta \to \infty) = 1.33(2D) > 0.88(3D)$. Therefore the influence of dimensionality d is an important factor on crossover physical parameters also.

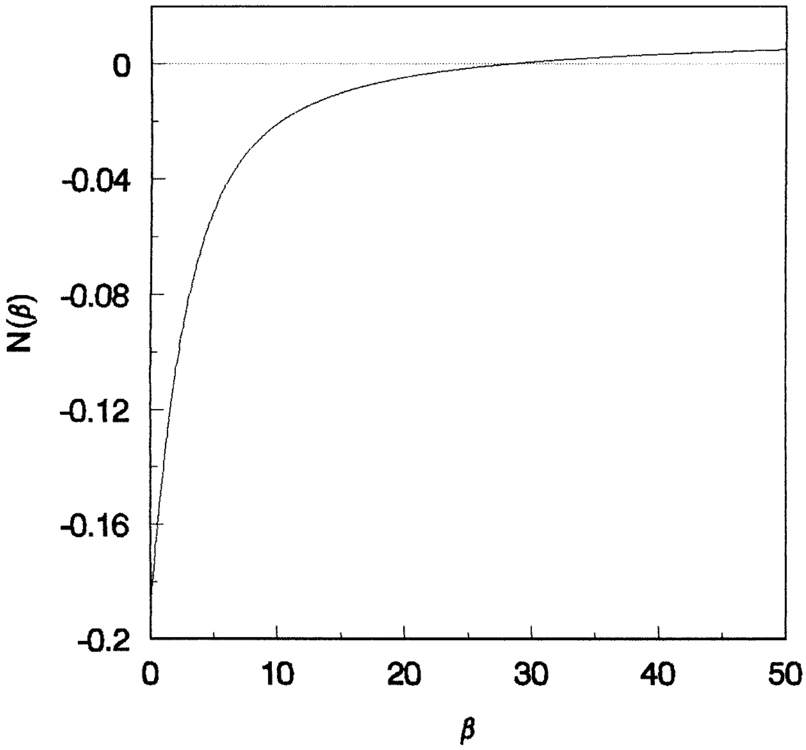

Figure 2. For a two-dimensional percolating S/N system, the critical exponent $N(\beta)$ of the crossover current density $|\vec{J}_{L-NL}|$ as a function of β based on equation (32).

As mentioned above, EMA can describe qualitatively the vanish of the crossover behaviour but gives incorrect exponents. For example, the EMA predicts that $M(\beta)$ and $N(\beta)$ are independent of β, while equations (40) and (41) give that $M(\beta)$ and $N(\beta)$ are monotonically increasing functions of β. The EMA predicts the divergence of crossover current density for all β, but equations (41) and (42) give that for large β $|\vec{J}_{L-NL}|$ can also vanish as $f \to f_c^-$. However, both EMA and equations (34)–(39) can predict the vanishing of the crossover field $|\vec{E}_{L-NL}|$ and the divergence of $|\vec{J}_{L-NL}|$ for $\beta < \beta_c$. So still EMA can be used as a first step to estimate critical exponents and to depict crossover properties of $|\vec{E}_{L-NL}|$ and $|\vec{J}_{L-NL}|$ near f_c.

2.4. Comparison with the bounds

Based on the so-called 'nodes–links–blobs' picture, Hui [13] gives the upper bounds for the crossover exponent of $|\vec{J}_{L-NL}|$. In this picture of the percolating cluster just below f_c, the conductance of the network is still finite, but typical size clusters of superconductors (TSCSs) exist. The TSCSs can be approximately denoted by an array of nodes separated by the correlation length ξ, each node is regarded as the centre of the superconducting cluster of linear size of order ξ and neighbouring clusters are linked by a thin layer of nonlinear normal conductors. In some places, there is only one normal bond separating the superconducting clusters. These bonds are called 'singly disconnected bonds' (SDBs); the number of SDBs diverges as $N_c \sim (f_c - f)^{-1}$ as $f \to f_c^-$.

As the voltage V_0 is applied to the network with volume $\Omega = L^d$, the voltage applied to the layer of normal bonds should be $\tilde{V} = (\xi/L)V_0$ and the current in each SDB is given by $\tilde{I} = [I/(L/\xi)^{d-1}]N_c$, if we neglect the effects of multiply connected regions. $\tilde{I}$ and $\tilde{V}$ must follow equation (1) [24], i.e.,

$$\tilde{I} = \sigma_1\tilde{V} + \chi_1\tilde{V}^{\beta+1}. \tag{43}$$

By simple deduction, we can obtain

$$\frac{\xi^{d-1}}{N_c}|\vec{J}| = \sigma_1\xi|\vec{E}| + \chi_1(\xi|\vec{E}|)^{\beta+1}. \tag{44}$$

Letting the magnitude of the linear part be equal to the nonlinear part, we have the crossover electric field

$$|\vec{E}_{L-NL}| = \left(\frac{\sigma_1}{\chi_1}\right)^{\frac{1}{\beta}}\xi^{-1} \sim (f_c - f)^{M(\beta)} \tag{45}$$

and the corresponding crossover current density

$$|\vec{J}_{L-NL}| \sim \frac{N_c}{\xi^{d-1}} \sim (f_c - f)^{N(\beta)} \tag{46}$$

with crossover exponents $M(\beta)$ and $N(\beta)$ given by

$$M(\beta) = \nu \qquad \text{and} \qquad N(\beta) = \nu(d-1) - 1. \tag{47}$$

It is known that the standard values of the correlation length exponent ν are 4/3 for $d = 2$ and 0.88 for $d = 3$. Thus, the upper bounds given by the NLB picture for the crossover exponents are $N(\beta) \simeq \frac{1}{3}$ for $d = 2$ and $N(\beta) \simeq 0.76$ for $d = 3$, which are independent of nonlinear exponent β; $M(\beta) \simeq \frac{4}{3}$ (2D) and $\simeq$0.88 (3D) for arbitrary β. The merit of the NLB picture lies in its simplicity and the bounds can be used as a limit to check the validity of the theoretical predictions. Comparing with them, it is not difficult to find that our results are always below the upper bounds. However, the bounds cannot describe how the effect of nonlinearity β acts on these crossover physical parameters. We think the upper bounds are applicable strictly for $\beta \to +\infty$.

The upper and lower bounds for $M(\beta = 2)$ have been reported also [14], such as $1.18 \leqslant M(2) \leqslant 1.33$ (2D) and $0.66 \leqslant M(2) \leqslant 0.88$ (3D). By means of the multifractal approach, we give $M(2) \approx 1.22$ (2D) and $M(2) \approx 0.75$, which is in good agreement with these bounds also.

3. Conclusions

In this work, we have studied the critical behaviour of the crossover electric field and current density in a percolating S/N random network by means of (i) EMA and (ii) the multifractal approach. The results are important and necessary for the investigation of the effective nonlinear response with arbitrary nonlinear exponent β. Below the percolation threshold of the superconductor, we have shown that the magnitudes of the crossover field $|\vec{E}_{L-NL}|$ and current density $|\vec{J}_{L-NL}|$ behave as $|\vec{E}_{L-NL}| \sim (f_c - f)^{M(\beta)}$ and $|\vec{J}_{L-NL}| \sim (f_c - f)^{N(\beta)}$. The EMA gives $M(\beta) = \frac{1}{2}$ and $N(\beta) = \frac{1}{2}$ for all nonlinearity β and dimension d. By means of the multifractal approach, we give $M(\beta) = \nu + [\zeta(\beta+2) - \zeta(2)]/\beta$ and $N(\beta) = (d-1)\nu + [\zeta(\beta+2) - (\beta+1)\zeta(2)]/\beta$. Both approximations predict $M(\beta) > 0$, which means that we can introduce $|\vec{E}_{L-NL}|$ to mark the transition where the effective linear and nonlinear response become equal in magnitude. This corresponds to the nonlinear response of the composite becoming more pronounced as the threshold is approached, and a small electric field is enough to lead to a considerable nonlinear response. At the same time, the second approximation predicts the monotonic increase of $M(\beta)$, thus for large β we may predict that a fairly small field is needed to stimulate an appreciable nonlinear response. $|\vec{J}_{L-NL}|$ takes such complex and interesting behaviour as to diverge, keep invariance or vanish with increasing β as $f \to f_c^-$. This implies that for $\beta > \beta_c$ we may also introduce the current $|\vec{J}_{L-NL}|$ to characterize the crossover behaviour, which was neglected in the percolating S/N system in the case of $\beta = 2$. Results are compared with those shown by the NLB picture and good agreement is found.

Although our discussions are only limited to a nonlinear $\vec{J}$–$\vec{E}$ relation, they can be readily generalized to systems on the effective mechanical properties in random mixtures. In the elastic percolation problem a 'superconductor' corresponds to a hard material and the nonlinear material corresponds to a medium with a stress–strain relation exhibiting nonlinear response for large strain. There is a linear response regime at small strain and a nonlinear regime at large strain [25]. As f_c is approached from below, the linear response regime corresponding to some measurable physical quantities such as bulk and shear modulus shrinks.

Our attention mainly concentrates on the crossover behaviour in a two-dimensional percolating S/N system. It would be worthwhile to study the crossover effects above two dimensions in which explicit expressions $\zeta(\beta + 2)$ have not yet been found. For a realistic system, the ratio of poor conductance to good conductance ($h \equiv g_1/g_2$) [9, 26, 27] may not be zero; we can take one step forward to discuss the crossover effect and scaling behaviour in such a system. It is interesting to perform numerical simulation on different nonlinear exponents β to study the crossover behaviour and verify the theoretical predictions shown here. Finally, we also hope that the present work will stimulate further experiments to observe the behaviour we have predicted.

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